§1 Connectivity

k any field, $X/\mathrm{Spec}\, k$ prop sm 1-dim geom connected.

Geometrically connected: $X_{\bar{k}}$ connected

Equivalently, $K := H^0(X, \mathcal{O}_X) = k$.

(Pf X proper $\Rightarrow$ K fin dim, X smooth $\Rightarrow$ K/k étale

$\bar{k}/k$ affine flat $\Rightarrow$ $H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) = \bar{k} \otimes_k K$

Thus $X_{\bar{k}}$ connected $\Leftrightarrow$ $\bar{k} \otimes_k K = \bar{k}^{[K:k]}$ no non-triv. idempot

$\Leftrightarrow$ $[K:k] = 1$.)

Try yourself: ECs are geom conn.

§2 Serre duality

Thm (Serre Duality, see Hartshorne §III.7)

X/k smooth projective purely d-dimensional, $\omega_X := \Lambda^d \Omega^1_{X/k}$.

Then $\exists\ \mathrm{tr}: H^d(X, \omega_X) \longrightarrow k$ s.th. $\forall$ loc. free $\mathcal{E}$ $\forall i$

$$H^i(X, \mathcal{E}) \times H^{d-i}(X, \mathcal{E}^\vee \otimes \omega_X) \longrightarrow H^d(X, \omega_X) \xrightarrow{\mathrm{tr}} k$$

is a perfect pairing.

For our smooth curve X, tr can be made explicit through residue formalism. Requires $k = \bar{k}$, but one may check that

$$\begin{array}{ccc} H^1(X_{\bar{k}}, \Omega_{X_{\bar{k}}}) & \xrightarrow{\mathrm{tr}} & \bar{k} \\ \cup\!| & & \cup\!| \\ H^1(X, \Omega_X) & \overset{\exists}{\dashrightarrow} & k \end{array}$$

(With non-trivial residue field extns, becomes more complicated, requires some form of Cohen structure Thm + usage of $\mathrm{tr}_{\kappa(x)/k}$.)

Consider ex seq (of qcoh sheaves)

$$0 \to \Omega_X \to \underline{\Omega_{X,\eta}} \to \underline{\Omega_{X,\eta}}/\Omega_X \to 0$$

Here $\Omega_{X,\eta}$:= stalk of Ω_X in gen pt η, is 1-dim $\mathcal{O}_{X,\eta}$-vsp.

$\underline{\Omega_{X,\eta}}$ = constant sheaf $U \mapsto \Omega_{X,\eta}$

$$\underline{\Omega_{X,\eta}}/\Omega_{X,\eta} \cong \bigoplus_{x \in X \text{ closed}} i_{x,*}\left(\Omega_{X,\eta}/\Omega_{X,x}\right) \qquad \cong \mathcal{O}_{X,\eta}/\mathcal{O}_{X,x}$$

Since $\underline{\Omega_{X,\eta}}$ flasque, $H^1(X, \underline{\Omega_{X,\eta}}) = 0$. Thus get

$$\Omega_{X,\eta} \to \bigoplus_x \Omega_{X,\eta}/\Omega_{X,x} \to H^1(X, \Omega_X) \to 0$$

Let $t \in \mathcal{O}_{X,x}$ be uniformizer. Any $\alpha \in \Omega_{X,\eta}$ may be written as $\left(\sum_{-n \le i \le -1} a_i t^i + h\right) dt$ w/ $a_i \in k$, $h \in \mathcal{O}_{X,x}$.

(This is where we use that residue field ext is trivial.)

<u>Def</u> $\mathrm{res}_x(\alpha) := a_{-1}$

<u>Prop</u> 1) Independent of choice of t.

2) $k = \bar{k}$. $\forall \alpha$ $\sum_{x \in X \text{ closed}} \mathrm{res}_x(\alpha) = 0$.

In other words, $\sum_{x \in X} \mathrm{res}_x(-)$ factors through $H^1(X, \Omega_X)$.

Thus defines trace.

Upshot $\forall\ \mathcal{E}$ vb on X, $H^i(X,\mathcal{E}) \simeq H^{1-i}(X, \mathcal{E}^\vee \otimes \Omega)^\vee$

canonically.

Cor $h^1(\Omega_X) = h^0(\mathcal{O}_X) = 1$ (put $\mathcal{E} = \mathcal{O}_X$)

$h^0(\Omega_X) = h^1(\mathcal{O}_X) = g$ (put $\mathcal{E} = \Omega_X$)

Def g is genus of X.

§ 3 Riemann – Roch

Recall Euler - Poincaré characteristic $\mathcal{F}$ coh. $\mathcal{O}_X$-module

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$$

Quick simple $\mathcal{F} = \mathcal{F}_{tors} \oplus \mathcal{F}_{loc\ free}$ $\quad \chi(\mathcal{F}) = h^0(\mathcal{F}_{tors}) + \chi(\mathcal{F}_{loc\ free})$

Example

$$\chi(\mathcal{O}_X) = 1 - g$$

$$\chi(\omega_X) = g - 1$$

$$\chi(i_{x,*}\kappa(x)) = [\kappa(x) : k].$$

Additive on ex seq: For $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$,

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G})$$

$\Longrightarrow$ Allows to compute $\chi(\mathcal{L})$ from $\chi(\mathcal{O}_X)$ $\forall$ lb. $\mathcal{L}$!

1) $\mathrm{Div}(X) := \bigoplus_{x \in X \text{ closed}} \mathbb{Z} \cdot [x]$. (Weil) divisors

2) $f \in k(X)^\times$ defines $\mathrm{div}(f) := \sum_x v_x(f) \cdot [x]$.

3) $\mathrm{Pic}(X) = \{ L \text{ lb on } X \} / \cong$

Abelian grp wrt. $\otimes$.

$$\begin{array}{ccc} \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) \\ & \searrow & \nearrow \cong \\ & \mathrm{Div}(X) / \mathrm{div}\, k(X)^\times & \end{array} , \quad D \longmapsto \mathcal{O}(D)$$

If $D = \sum n_x \cdot x$, $\mathcal{O}(D)(U) = \{ f \in k(X) \mid v_x(f) \geq -n_x \ \forall x \in U \}$

(ie pole order at $x \leq n_x$)

4) <u>Degree of a divisor</u> $\deg : \mathrm{Div}\, X \to \mathbb{Z}$

$$\sum n_x [x] \longmapsto \sum n_x [\kappa(x) : k]$$

Then $\deg(\mathrm{div}(f)) = \sum_{x \in f^{-1}(0)} e_x [\kappa(x) : k] - \sum_{x \in f^{-1}(\infty)} e_x [\kappa(x) : k]$

where f viewed as $X \to \mathbb{P}^1$.

Lecture III §1: $\deg(\mathrm{div}(f)) = 0$.

$\Longrightarrow \deg : \mathrm{Pic}(X) \to \mathbb{Z}$.

Thm (Riemann Roch) L lb on X. Then

$$\chi(L) = \deg L + \underbrace{1-g}_{=\ \chi(\mathcal{O})}.$$

Cor $\deg \Omega_X = \chi(\Omega_X) + g - 1$

$= 2g-2$

§4 ECs have genus 1

Prop E/k EC. Then $g(E) = 1$.

Pf To show: $h^0(\Omega_E) = 1$. Next Prop shows $\Omega_E \cong \mathcal{O}_E$, which implies claim by geometric connectedness. □

Prop $p: G \to \operatorname{Spec} k$ group scheme

Then $\Omega^1_{G/k} \cong p^* e^* \Omega^1_{G/k}$.

Idea Every differential form $\omega_0 \in e^* \Omega^1_{G/k}$ ($\operatorname{Spec} k \xrightarrow{e} G$ neutral elt) extends in a unique way to left-translation-invariant diff-form on all of G.

(Same w/ right-invariant.)

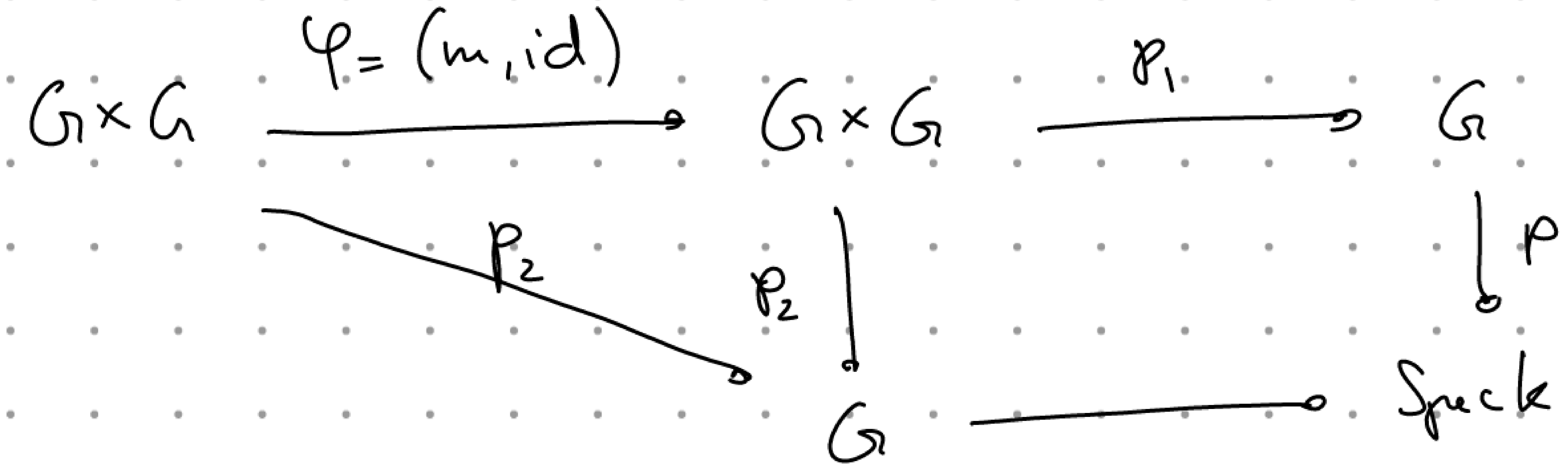
Example (Lie groups) $f(t)\,dt$ on $\mathbb{R}^\times$ is left-invariant if

$\forall a \in \mathbb{R}^\times$, $f(at)\,d(at) = a f(at)\,dt = f(t)\,dt$.

$\iff f(t) = \frac{f(1)}{t}$ i.e. diff form $=$ const. $\frac{dt}{t}$.

dz on any analytic EC $\mathbb{C}/\Lambda$.

Proof of Prop. Consider



Then $p_1^* \Omega^1_{G/k} = \Omega^1_{G\times G/G} \xleftarrow[\gamma]{\sim} \varphi^* \Omega^1_{G\times G/G} = \varphi^* p_1^* \Omega^1_{G/k}$.

Apply $(e, id)^*$ to this identity.

On LHS get $p^* e^* \Omega^1_{G/k}$, on RHS $\Omega^1_{G/k}$. □

Example $\mu_p = \mathrm{Spec}\, A$, $A = \mathbb{F}_p[t]/_{t^p - 1}$

$\Omega^1_{A\otimes A/A} = (A \otimes A) dx$, $\varphi^*(dx) = d(xy) = y\,dx$

$x = t \otimes 1$, $y = 1 \otimes t$

(Caution: Working over G via p_2, not over $\mathrm{Spec}\, k$, hence $dy = 0$.)

Thus $\gamma^{-1}(dx) = y^{-1} dx$.

So starting with $dt \in p^* e^* \Omega^1_{\mu_p/\mathbb{F}_p}$,

view as $dx \in \Omega^1_{\mu_p \times \mu_p / \mu_p}$

$\xmapsto{\gamma^{-1}} y^{-1} dx = t^{-1} dt$ under $(p_1 \circ \varphi \circ (e, id)) = id_G$ □

Do yourself: Work out for $\mathbb{G}_a$, $\mathbb{G}_m$.

§ 5 ECs as Cubics

Prop $\exists$ embedding $E \hookrightarrow \mathbb{P}^2_k$, realizing it as a smooth cubic.

Recall Given $\mathcal{L}$ lb on $X/_k$ + $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$

s.th. $\mathcal{O}_X^{\oplus n+1} \xrightarrow{s_i} \mathcal{L}$ surjective, (in partic $\mathcal{L}$ globally generated)

get morphism $\varphi_s : X \longrightarrow \mathbb{P}^n_k$ s.th. $\varphi_s^* \mathcal{O}(1) \cong \mathcal{L}$.

Description: $X = \bigcup_{i=0}^{n} D(s_i)$.

Then $\varphi_s|_{D(s_i)} : D(s_i) \longrightarrow D_+(T_i) \cong \mathbb{A}^n$

$$\Gamma(X, \mathcal{O}_X) \ni \frac{s_j}{s_i} \longleftarrow\!\shortmid \frac{T_j}{T_i} \in \Gamma(D_+(T_i))$$

Lem X/k proper normal separated curve. $\mathcal{L}$ on X lb.

1) $\mathcal{L}$ glob gen $\Leftrightarrow$ codim $\Gamma(\mathcal{L}(-x)) \subseteq \Gamma(\mathcal{L})$ is 1 $\forall x$

(ie $\forall x \; \exists \; s \in \Gamma(\mathcal{L})$ non-vanishing at x) = generating $\mathcal{L}$ at x

2) Map induced from k-basis of $\Gamma(\mathcal{L})$

closed immersion $\Leftrightarrow$ codim $\Gamma(\mathcal{L}(-x-y)) \subseteq \Gamma(\mathcal{L})$ is 2

$\forall x, y$, possibly $x = y$.

Proof as exercise.

(Hint for 2): $x \neq y$ - case gives injectivity on underlying top spaces
$x = y$ - case gives surjectivity at local rings from Nakayama.)

$\square$

Lem $X/\mathrm{Spec}\,k$ geom conn, smooth prop. genus 1 curve.

1) $\deg \mathcal{L} < 0 \Rightarrow h^0(\mathcal{L}) = 0$

2) $\deg \mathcal{L} = 0 \Rightarrow h^0(\mathcal{L}) \neq 0 \Leftrightarrow \mathcal{L} \cong \mathcal{O}$.

3) $\deg \mathcal{L} > 0 \Rightarrow h^0(\mathcal{L}) = \deg \mathcal{L}$.

Pf 1) always true, since $h^0(\mathcal{L}) > 0 \Leftrightarrow \exists\, \mathcal{O} \xrightarrow{\neq 0} \mathcal{L}$.

Since X integral, injective. Then $\mathcal{L}$ defined by effective divisor defined from torsion sheaf $\mathcal{L}/\mathcal{O}$.

2) Similar: $\exists\, \mathcal{O} \xrightarrow{\neq 0} \mathcal{L} \Rightarrow \deg \mathcal{L} = \deg \mathcal{O} + \mathrm{len}_k\, \mathcal{L}/\mathcal{O}$

$= \deg \mathcal{O}$

$\Rightarrow$ is iso.

3) This requires genus 1 assumption.

$\chi(\mathcal{L}) = \deg \mathcal{L}$ (since $1-g=0$) by R.-R.

Serre duality gives $h^1(\mathcal{L}) = h^0(\Omega^1 \otimes \mathcal{L}^\vee)$.

by 1) + (Genus 1 $\Rightarrow$ $\deg \Omega^1 = 2g-2 = 0$) $\square$

<u>Proof that any $E \hookrightarrow \mathbb{P}^2$ a cubic.</u>

E elliptic $\Rightarrow \exists\, e \in E(k)$, hence lb of deg 1

$$\mathcal{L} := \mathcal{O}(e).$$

$$\Rightarrow \exists \text{ lb of deg } 3,\ \mathcal{L}^{\otimes 3}$$

Then $h^0(\mathcal{L}^{\otimes 3}) = 3$ by Lemma

& $h^0(\mathcal{L}^{\otimes 3}(-x-y)) = 1$ by Lemma.

$\Rightarrow$ General Lemma before : $E \hookrightarrow \mathbb{P}(\Gamma(\mathcal{L}^{\otimes 3})^\vee)$.

<u>Lem</u> $Y = V(F) \subseteq \mathbb{P}^2_k$ defined by homogeneous poly of deg d (= section $\neq 0$ of $\mathcal{O}(d)$).

Then Y is connected, 1-dimensional, $h^1(Y, \mathcal{O}_Y) = \frac{(d-1)(d-2)}{2}$

<u>Sketch</u> Know

$$H^i(\mathcal{O}(n)) = \begin{cases} k[T_0, T_1, T_2]_{\deg = n} & i = 0 \\ 0 & i = 1 \\ \left((T_0T_1T_2)^{-1} k[T_0^{-1}, T_1^{-1}, T_2^{-1}]\right)_{\deg = n} & i = 2. \end{cases}$$

Apply this to the colour seq for

$$0 \to \mathcal{O}(-d) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Y \longrightarrow 0 \quad \square$$